Universität Konstanz
Fachbereich Mathematik und Statistik
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Exercises for the course "Linear Algebra I"

Sheet 11

Hand in your solutions on Thursday, 23. Januar 2020, 09:55, in the Postbox of your Tutor in F4. Please try to write your solutions clear and readable, write your name and the name of your tutor on each sheet, and staple the sheets together.

Exercise 11.1 ***Beweismechanikaufgabe*** *(4 points)*

Bitte gehen Sie in dieser Aufgabe nach den Regeln der Beweismechanik vor und geben Ihre Lösung auf einem separaten Blatt in den Briefkasten mit der Aufschrift „Beweismechanikaufgaben" ab. Ihnen unbekannte Begriffe und Symbole können Sie in der Beweismechanik nachschlagen.

Sei V ein $\mathbb{R}$-Vektorraum und $U, W \subset V$ zwei nichtleere Unterräume von V. Zeigen Sie, dass folgende Aussagen äquivalent sind:

(i) $U \cap W = \{0\}$

(ii) Für alle $d, r \in \mathbb{N}$ gilt: Ist $A = \{\alpha_1, \ldots, \alpha_d\} \subset U$ linear unabhängig und $B = \{\beta_1, \ldots, \beta_r\} \subset W$ linear unabhängig, so ist auch $A \cup B$ linear unabhängig.

Exercise 11.2 *(3 points)*

Let K be a field and $f = \sum_{i=0}^m a_i X^i$, $g = \sum_{i=0}^n b_i X^i \in K[X]$ polynomoials over K. We define the product of f and g as follows:

$$f \cdot g = \sum_{k=0}^{m+n} \left(\sum_{i+j=k} a_i b_j \right) X^k,$$

where $\sum\limits_{i+j=k}$ denotes the sum over all pairs $(i, j) \in \mathbb{N}_0 \times \mathbb{N}_0$ such that $i + j = k$ applies.

In the following you may without proof assume that $K[X]$ is a K-vector space and that the product $\cdot$ is associative.

(a) Show that $(K[X], +, \cdot)$ is a commutative ring with 1.

(b) Show that $(K[X], +, \cdot)$, along with the scalar multiplication from Excercise 8.1, is a linear algebra over K.

Exercise 11.3 *(5 points)*

Let K be a field and $m, n \in \mathbb{N}$.

(a) For a matrix $A \in \mathrm{Mat}_{m\times n}(K)$ we define the map $U_A : K^m \to K^n$ via $U_A(\alpha) := \alpha A$. Show that U_A is a linear map for any matrix A. Moreover, determine the matrix representation (*Matrixdarstellung*) of U_A with respect to the standard bases of K^m and K^n, respectively (cf. Beispiel 20.4(2))

(b) Let $\mathcal{B}_1 := \{(0, 2, 1), (0, 1, 2), (1, 0, 1)\}$ and $\mathcal{B}_2 := \{(-1, 1, 1), (-2, 0, 3), (0, 3, 2)\}$. Show that $\mathcal{B}_1$ and $\mathcal{B}_2$ are bases of $\mathbb{R}^3$ and determine the matrix representation of the identity function $\mathrm{id} : \mathbb{R}^3 \to \mathbb{R}^3$, defined by $\mathrm{id}(x) = x$ for all $x \in \mathbb{R}^3$, with respect to the ordered bases $\mathcal{B}_1$ and $\mathcal{B}_2$.

Exercise 11.4 *(4 points)*

Let K be a field.

(a) Let $\pi : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ be a bijective map and let $f_\pi : K^{n\times 1} \to K^{n\times 1}$ be defined by

$$f_\pi\left(\begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix}\right) = \begin{pmatrix} x_{\pi(1)} \\ x_{\pi(2)} \\ \ldots \\ x_{\pi(n)} \end{pmatrix}.$$

Determine the matrix representation of f_π with respect to the standard basis of $K^{n\times 1}$.

(b) Let $\mathbb{B} = (v_1, v_2, v_3, v_4)$, where $v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ and $v_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. Determine an invertible matrix P such that for all $\alpha \in K^{4\times 1}$ the following holds: $P[\alpha]_{\mathcal{E}} = [\alpha]_{\mathbb{B}}$, where $\mathcal{E}$ denotes the standard basis of the K-vector space $K^{4\times 1}$.